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1982 J. Phys. A: Math. Gen. 15 1705

(http://iopscience.iop.org/0305-4470/15/5/031)

J. Phys. A: Math. Gen. 15 (1982) 1705–1714. Printed in Great Britain

Tricritical susceptibility of scalar systems near four dimensions

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Received 3 September 1981

Abstract. Tricritical scaling is studied in a single-component Landau–Ginzburg–Wilson model of spatial dimensionality $d = 4 - \varepsilon$. At first order in ε, a single expression for the order-parameter susceptibility is exhibited, which behaves correctly in all limits and is uniformly applicable throughout the tricritical region. This is achieved by means of a renormalisation group scheme in which couplings depend explicitly on the order parameter and all three loci of ordinary critical points are controlled by a single fixed point.

1. Introduction

In the vicinity of a multicritical point, there occur several distinct, competing types of critical behaviour associated with the multicritical point itself, and with the various critical loci which emerge from it. Within the renormalisation group approach, it is usually straightforward to obtain both the leading singularities and singular corrections to the leading scaling behaviour, associated with each type of critical behaviour separately. It is more difficult to construct explicitly expressions for thermodynamic functions which correctly exhibit all the singularities in appropriate limits.

The purpose of this paper is to display such an expression for the order-parameter susceptibility in the neighbourhood of a tricritical point (Griffiths 1970). In the case of a tricritical point, two special problems occur. Firstly, the borderline dimensionality, above which a mean-field-like description is adequate, has the value $d^{\times} = 3$ for the tricritical point (Riedel and Wegner 1972) and $d^{\times} = 4$ for each of the three critical loci which meet there. (Readers who are unfamiliar with the tricritical phase diagram may like to consult the article by Lawrie (1979, referred to hereafter as I) for an elementary introduction.) This reflects the necessity of including, in an effective Hamiltonian of the Landau–Ginsburg–Wilson type, the operator ϕ^6, in order to maintain thermodynamic stability in the tricritical region. Near the loci of ordinary critical points, this operator gives rise to corrections, which must appear in the correct scaling form, in addition to those arising directly from crossover associated with the operator ϕ^4. By contrast, a bicritical point, for example, has the same borderline, $d^{\times} = 4$, as the critical loci, and there the problem reduces asymptotically to calculating crossover scaling functions which depend only on a single scaling variable (Pfeuty *et al* 1974, Amit and Goldschmidt 1978). Secondly, in the model we consider, two of the critical loci bound 'wing' coexistence surfaces which are symmetrically disposed about a symmetry plane containing the tricritical point. In the neighbourhood of one of these loci, where renormalisation group calculations are most easily carried out, the full symmetry of the phase diagram is not apparent, but this symmetry must be respected by any expression which is to describe the whole tricritical region.

In earlier work reported in I, it was shown that the additional operator ϕ^6 could be systematically included in the framework of renormalised perturbation theory. This is not entirely trivial, since the exponents of the critical loci, and therefore also the scaling functions, must be calculated by means of an ε expansion about four dimensions, where ϕ^6 is non-renormalisable As a consequence, additional higher-order operators (indeed, an infinite number of them) are required to remove all ultraviolet divergences. It was argued, however, that, at any finite order of the ε expansion, only a finite number of these operators are explicitly required, provided that the additional corrections to scaling which they induce are ignored. The fixed points of the renormalisation group which control behaviour near the wings were located for the first time. For spatial dimensionality in the range $3<d=4-\varepsilon<4$ (to which this work is also restricted), a scaling form of the equation of state was obtained which describes crossover from each critical locus to the tricritical point. This was achieved, however, at the expense of concentrating on the neighbourhood of one locus at a time, with a consequent loss of the full symmetry.

To remove this deficiency, we present here a modified renormalisation group for the single-component model, using effective coupling constants which depend explicitly on the order parameter M. In this scheme, the singularities at all three critical loci are controlled by a single Ising-like fixed point, and the symmetry is maintained at each stage. This scheme is described in § 2. We have not found it possible to construct the equation of state analytically by this method, and we concentrate instead on the susceptibility, which is obtained in § 3.

2. Renormalisation of a field theory model

The model we consider is defined by the effective reduced Hamiltonian density

$$\mathscr{H}=-H\phi+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}r_0\phi^2+\frac{1}{4!}u_0\phi^4+\frac{1}{6!}v_0\phi^6+\Delta\mathscr{H} \tag{2.1}$$

where $\Delta\mathscr{H}$ denotes the set of higher-order operators and their coupling constants which, as discussed in § 1, are required to effect a consistent renormalisation program. This model lacks a term proportional to ϕ^3, which would be required for a full description of tricriticality, but should be adequate to represent symmetrical systems such as metamagnets (Nelson and Fisher 1975). In the case of an Ising-like metamagnet, $\phi(x)$ may be thought of as a staggered single-component spin density, and H as its conjugate staggered magnetic field. The remaining parameters then depend smoothly on temperature and on a uniform magnetic field, although, in the tricritical region, v_0 may be taken as a positive constant. Zero subscripts denote unrenormalised quantities; they have been omitted from H and ϕ because no wavefunction renormalisation is required at the one-loop level we consider explicitly.

In the symmetry plane, $H=0$, one critical locus, the lambda line, is located by the condition $r_0=r_{0c}(u_0, v_0)$ for which the inverse susceptibility vanishes, and the tricritical point $u_0=u_{0t}(v_0)$ by the simultaneous vanishing of the four-point vertex function at zero wavevector. The actual values of r_{0c} and u_{0t} play no significant role and, within the dimensional regularisation scheme which we adopt, may consistently be taken as zero. As described in detail in I, all ultraviolet divergences may be eliminated at one-loop order, by introducing renormalised parameters t, u and v, corresponding respectively to r_0, u_0 and v_0, and by taking

$$\Delta\mathcal{H} = \frac{1}{8!} w_0 \phi^8 \tag{2.2}$$

where w_0 is determined in terms of the remaining parameters, and introduces no additional parametric dependence. As usual, the scheme involves an arbitrary parameter μ, with the dimensions of inverse length, which may be used to make t, u and v dimensionless.

Away from the symmetry plane, with $H \neq 0$, these parameters do not provide a convenient description. It is then useful to exchange H for a staggered magnetisation variable M, by making the shift

$$\phi(x) = M + \psi(x) \tag{2.3}$$

and choosing M to ensure $\langle\psi\rangle = 0$. One then obtains a new set of unrenormalised parameters R_0, H_{30}, U_0, U_{50}, v_0 which multiply successive powers of ψ. From the explicit calculations given in I, we obtain the following relations between renormalised and unrenormalised quantities:

$$\mu^{-2} R_0 = \tau(1 + \tfrac{1}{2}UK) + \tfrac{1}{2}H_3^2 K \tag{2.4}$$

$$\mu^{-(1+\varepsilon/2)} H_{30} = H_3(1 + \tfrac{3}{2}UK) + \tfrac{1}{2}U_5 \tau K \tag{2.5}$$

$$\mu^{-\varepsilon} U_0 = U(1 + \tfrac{3}{2}UK) + \tfrac{1}{2}v\tau K + 2U_5 H_3 K \tag{2.6}$$

$$\mu^{1-3\varepsilon/2} U_{50} = U_5(1 + 5UK) + \tfrac{5}{2}vH_3 K \tag{2.7}$$

$$\mu^{2-2\varepsilon} v_0 = v(1 + \tfrac{15}{2}UK) + 5U_5^2 K \tag{2.8}$$

with higher-order counterterms given by

$$\mu^{3-5\varepsilon/2} U_{70} = \tfrac{35}{2} U_5 v K \tag{2.9}$$

$$\mu^{4-3\varepsilon} w_0 = \tfrac{35}{2} v^2 K \tag{2.10}$$

$$K = 2\pi^{d/2}/(2\pi)^d \Gamma(\tfrac{1}{2}d)\varepsilon. \tag{2.11}$$

The M-dependent renormalised parameters are related to the M-independent quantities t, u, v by

$$\tau = t + \tfrac{1}{2}uM^2 + \frac{v}{4!}M^4 \tag{2.12}$$

$$H_3 = uM + \frac{v}{3!}M^3 \tag{2.13}$$

$$U = u + \tfrac{1}{2}vM^2 \tag{2.14}$$

$$U_5 = vM \tag{2.15}$$

where M has been rendered dimensionless by extraction of $\mu^{1-\varepsilon/2}$. The new variables U_{50} and U_5 are clearly not independent of the remaining parameters, since the original Hamiltonian (2.1) contains only the four parameters H, r_0, u_0, v_0. However, the operator ψ^5 scales with an independent exponent Ψ_5, and to obtain this behaviour correctly, it is essential to treat U_5 as an independent variable within the renormalisation group scheme. It is important to realise, therefore, that the relations (2.4)–(2.10) are in fact sufficient to renormalise the theory with *five* independent parameters which would result from removing the constraint (2.15). Our procedure will be to write down and solve a renormalisation group equation involving all five independent parameters, and only when this is done to reintroduce the relations (2.12)–(2.15).

The equation in question may be written in the form

$$\left(A_2\frac{\partial}{\partial\tau}+A_3\frac{\partial}{\partial H_3}+W\frac{\partial}{\partial U}+A_5\frac{\partial}{\partial U_5}+A_6\frac{\partial}{\partial v}+\gamma\right)\chi^{-1}=0 \tag{2.16}$$

where χ denotes the order-parameter susceptibility. Normally, such an equation embodies the statement that, when expressed in terms of unrenormalised quantities, the thermodynamic functions are independent of μ. On this basis, the coefficients can be determined by differentiation of (2.4)–(2.8) with respect to μ at fixed $R_0, \ldots, v_0$, with the results

$$A_2=-\tau+\varepsilon H_3^2/6u^* \tag{2.17}$$

$$A_3=-\tfrac{1}{2}H_3(1+\tfrac{1}{2}\varepsilon-5\varepsilon U/6u^*)+\{\varepsilon\tau U_5/6u^*\} \tag{2.18}$$

$$W=-\tfrac{1}{2}\varepsilon U(1-U/u^*)+\{\varepsilon\tau v/6u^*+2\varepsilon H_3U_5/3u^*\} \tag{2.19}$$

$$A_5=\tfrac{1}{2}U_5(1-3\varepsilon/2+7\varepsilon U/2u^*)+\{5\varepsilon H_3 v/6u^*\} \tag{2.20}$$

$$A_6=v(1-\varepsilon+8U/3u^*)+\{5\varepsilon U_5^2/6u^*\} \tag{2.21}$$

$$\gamma=1+\varepsilon U/6u^*. \tag{2.22}$$

In these equations, the fixed-point coupling constant is

$$u^*=2/3K, \tag{2.23}$$

and braces indicate terms which require special treatment in the next section. Unfortunately, in the presence of the counterterms (2.9) and (2.10), the unrenormalised theory is not independent of μ. It was argued in I that the renormalisation group equation may nevertheless be used in the normal way, provided that one ignores higher-order corrections to scaling associated with the operators ψ^7, ψ^8, etc, which are not properly included. In particular, critical exponents associated with the Ising-like fixed point $U=u^*$ are obtained by omitting the final terms of (2.18), (2.20) and (2.21), and are given by

$$\beta=\tfrac{1}{2}(1-\varepsilon/3) \tag{2.24}$$

$$\Psi_5=-\tfrac{1}{2}(1+2\varepsilon) \tag{2.25}$$

$$\Psi=-(1+5\varepsilon/3) \tag{2.26}$$

$$\gamma=1+\varepsilon/6 \tag{2.27}$$

at this order, in agreement with standard results.

3. Susceptibility in the tricritical region

Before presenting our detailed calculation of the order-parameter susceptibility, we review briefly the scaling properties which are to be expected on general grounds. The critical loci where singularities occur are located by the vanishing of the two-and three-point vertex functions at zero wavevector. At one-loop order, and possibly at all orders in a suitable renormalisation scheme, this occurs when τ and H_3 vanish. The loci are then given by

$$\text{lambda line:} \quad t_\lambda = H_\lambda = M_\lambda = 0 \qquad u > 0 \tag{3.1}$$

$$\text{wings:} \quad t_c = 3u^2/2v \qquad M_c^\pm = \pm(-6u/v)^{1/2} \qquad H_c^\pm = (4u^2/5v)M_c^\pm \qquad u < 0 \tag{3.2}$$

and meet at the tricritical point $u = 0$. The Ising-like singularities at these loci were found in I to be controlled by fixed points at $u = u^*$ and $u = -\frac{1}{2}u^*$ respectively. Inspection of (2.14), (3.1) and (3.2) reveals that all three fixed points correspond to

$$U = u^*. \tag{3.3}$$

For $3 < d < 4$, tricritical behaviour is associated with the Gaussian fixed point $U = u = 0$, but, as explained in I, the mean-field-like tricritical exponents

$$\beta_t = \tfrac{1}{4} \qquad \gamma_t = 1 \qquad \phi_t = \tfrac{1}{2} \tag{3.4}$$

are not given directly by the renormalisation group.

Throughout the tricritical region, the susceptibility should be expressible in the scaling form

$$\chi^{-1} = |t|^{\gamma_t} f(M|t|^{-\beta_t}, u|t|^{-\phi_t}, v) \tag{3.5}$$

which is trivial in the classical approximation $\chi^{-1} = \tau$. Near the lambda line, for small t and M, we should have

$$\chi^{-1} \approx |t|^{\gamma} f_\lambda(M|t|^{-\beta}, (u-u^*)|t|^{-\omega}, v|t|^{-\Psi}) \tag{3.6}$$

with exponents given by (2.24), (2.26), (2.27) and

$$\omega = -\tfrac{1}{2}\varepsilon + O(\varepsilon^2). \tag{3.7}$$

Near the wings, when $\dot{t} = t - t_c$ and, say, $M = M - M_c^+$ are small, extra corrections appear due to the lack of symmetry, and we expect

$$\chi^{-1} \approx |t|^{\gamma} f_w(\dot{M}|\dot{t}|^{-\beta}, (u^* + 2u)|\dot{t}|^{-\omega}, \dot{u}_5|\dot{t}|^{-\Psi_5}, v|\dot{t}|^{-\Psi}) \tag{3.8}$$

with

$$\dot{u}_5 = U_5(v, M_c^+) = vM_c^+ \tag{3.9}$$

and the exponent Ψ_5 given by (2.25). All the scaling functions f_i should be analytic in their arguments. Those appearing in (3.6) and (3.8) exhibit the singular behaviour expected in the immediate vicinity of the critical loci; additional, analytic corrections may in general appear in intermediate regions described by (3.5).

We wish to derive an expression for the scaling function f in (3.5), which reduces to (3.6) or (3.8) in appropriate limits. This is accomplished by solving the renormalisation group equation (2.16) with a boundary condition given by direct perturbative calculation as

$$\chi^{-1} = \langle \hat{\psi}(q)\hat{\psi}(-q)\rangle_{q=0}^{-1} = \tau + \frac{\varepsilon}{6u^*}[u\tau \ln \tau + H_3^2(1 + \ln \tau)] \tag{3.10}$$

where $\hat{\psi}(q)$ is the Fourier transform of $\psi(x)$. As usual, solution by the method of characteristics yields the relation

$$\chi^{-1}[U_i] = \exp\left(\int_1^\lambda \frac{d\lambda'}{\lambda'}\gamma[\bar{U}(\lambda')]\right)\chi^{-1}[\bar{U}_i(\lambda)] \tag{3.11}$$

where, for brevity, U_i denotes the set of parameters τ, H_3, U, U_5, v and, if their respective coefficients in (2.16) are denoted by A_i, the characteristic functions of the free parameter λ are defined as solutions of

$$\lambda\frac{\partial\bar{U}_i}{\partial\lambda} = A_i[\bar{U}_j] \qquad \bar{U}_i(1) = U_i. \tag{3.12}$$

The various critical exponents arise from exponentiation of the logarithms in (3.10), to which end we fix the value of λ by the condition

$$\bar{\tau}(\lambda) = 1, \tag{3.13}$$

giving

$$\chi^{-1}[\bar{U}_i] = \bar{\tau} + (\varepsilon/6u^*)\bar{H}_3^2. \tag{3.14}$$

Solution of the characteristic equations (3.12), with A_i given by (2.17)–(2.21), is complicated by the inhomogeneous terms indicated by braces, and we have not found it possible to obtain exact solutions. To obtain approximate solutions, we observe that the inhomogeneous terms are all of one-loop order, and that solutions of the truncated equations obtained by omitting them already contain the correct exponents (2.24)–(2.26). The following procedure is therefore consistent both with the expected scaling behaviour and with our one-loop approximation: we first solve the truncated equations then substitute the approximate solutions into the inhomogeneous terms, and finally resolve the full equations to find improved approximations. In carrying out the first step, it is possible to retain the whole of equation (2.17). This is desirable, because critical singularities should occur only when τ and H_3 are simultaneously equal to zero. The solutions obtained in this way are

$$\bar{\tau} = \lambda^{-1}[\tau + (H_3^2/2U)(X^{2/3} - 1)] \tag{3.15}$$

$$\bar{H}_3 = \lambda^{-(1+\varepsilon/2)/2}X^{5/6}[H_3 + (\tau U_5/5U)(X^{5/3} - 1)] \tag{3.16}$$

$$\bar{U} = \lambda^{-\varepsilon/2}X[U + (4H_3U_5/7U)(X^{7/3} - 1) + (\tau v/10U)(X^{10/3} - 1)] \tag{3.17}$$

$$\bar{U}_5 = \lambda^{(1-3\varepsilon/2)/2}X^{7/2}[U_5 + (H_3v/U)(X^{5/3} - 1)] \tag{3.18}$$

$$\bar{v} = \lambda^{1-\varepsilon}X^{16/3}[v + (5U_5^2/U)(X^{2/3} - 1)] \tag{3.19}$$

with crossover controlled by the function

$$X(\lambda, U) = [1 + (U/u^*)(|\lambda|^{-\varepsilon/2} - 1)]^{-1}. \tag{3.20}$$

For future reference, we note that X may be expanded as

$$X = 1 + O(1\text{-loop}). \tag{3.21}$$

With the same approximation, the prefactor in (3.11) may be written as

$$\lambda X^{1/3}. \tag{3.22}$$

We may now use (3.11), (3.13) and (3.14) to express the inverse susceptibility as

$$\chi^{-1} = \lambda X^{1/3}\{1 + (\varepsilon/6u^*)\lambda^{-(1+\varepsilon/2)}X^{5/3}[H_3 + (\tau U_5/5U)(X^{5/3} - 1)]^2\} \tag{3.23}$$

with λ determined as the solution of

$$\lambda = \tau - H_3^2/2U + (H_3^2/2U)X^{2/3}. \tag{3.24}$$

The original parameters t, u, v and M may now be reintroduced via (2.12)–(2.15), and one sees explicitly that χ is a function only of M^2, maintaining the full symmetry of the phase diagram.

While (3.23) is satisfactory for negative u, it fails to satisfy (3.6) near the lambda line, since the exponent Ψ does not appear correctly. However, a simple modification suffices to ensure correct behaviour in all limits. Guided by the appearance of $\bar{\tau}$ and $\bar{U}_5$ in (3.11), we replace τ and U_5 in (3.23) by the expressions in square brackets in (3.15) and (3.18). The expansion (3.21) shows that the terms which this modification adds to χ^{-1} are effectively of two-loop order, and the modification is thus consistent with our one-loop approximation. Our final result then reads

$$\chi^{-1} = \lambda X^{1/3}[1 + (\varepsilon/6u^*)\lambda^{-(1+\varepsilon/2)}X^{5/3}Y^2] \tag{3.25}$$

with

$$Y = H_3 - (\tau U_5/5U) + (1/5U)X^{5/3}\lambda[(U_5 - H_3v/U) + (H_3v/u)X^{5/3}]. \tag{3.26}$$

We now verify that this expression behaves correctly near the critical loci. Consider first the neighbourhood of the lambda line, where u is positive, and t and M are small. We expand τ, H_3 and U in powers of t and M^2, retaining only the leading terms, with the results

$$\tau - H_3^2/2U \approx t \qquad H_3^2/2U \approx uM^2 \tag{3.27}$$

$$U_5 - H_3v/U \approx 0 \tag{3.28}$$

$$H_3v/U \approx vM. \tag{3.29}$$

Evidently, λ is also small in this region, and (3.24) yields

$$\lambda \approx t + uM^2(u\lambda^{\varepsilon/2}/u^*)^{2/3}. \tag{3.30}$$

Writing

$$\lambda = t\bar{\lambda} \qquad x = u(u^*/u)^{2/3}M^2t^{-2\beta} \tag{3.31}$$

with β given by (2.24) we have

$$\bar{\lambda} \approx 1 + x\bar{\lambda}^{\varepsilon/3}, \tag{3.32}$$

so that $\bar{\lambda}$ is analytic in x. With these approximations, (3.26) may be written as

$$Y \approx uM[1 + \bar{\lambda}^{-\Psi}(vt^{-\Psi})] \tag{3.33}$$

where a power of u and some numerical factors have been absorbed into v. Up to an overall factor, the susceptibility may then be written as

$$\chi^{-1} \approx t^{\gamma}\bar{\lambda}^{\varepsilon/6}\{1 + (\varepsilon/6)x\bar{\lambda}^{-2\beta}[1 + \bar{\lambda}^{-\Psi}(vt^{-\Psi})]^2\} \tag{3.34}$$

which has the required form (3.6), with all non-universal quantities absorbed into the scaling variables. Additional singular corrections involving $(u - u^*)|t|^{\varepsilon/2}$ arise from the expansion of X, whose leading term was retained in (3.30), and analytic corrections arise from the terms neglected in (3.27)–(3.29).

Near the wings, a similar analysis leads to a function of the form (3.8). Corrections due to the operator ψ^5 arise from taking

$$U_5 - H_3 v/U \approx vM_c^{\pm} = \pm \dot{u}_5 \tag{3.35}$$

in place of (3.28).

Thus, the expression (3.25) does indeed have the correct limiting forms near all three critical loci. It cannot, however, be written in the form (3.5). This difficulty arose also in I, where it was found that the equation of state had the formal appearance of scaling, but with incorrect Gaussian tricritical exponents

$$\beta_0 = \tfrac{1}{2}(1 - \tfrac{1}{2}\varepsilon) \qquad \gamma_0 = 1 \qquad \phi_0 = \tfrac{1}{2}\varepsilon \qquad \Psi_0 = -(1-\varepsilon). \tag{3.36}$$

It was pointed out that, quite apart from the theoretical prejudice expressed by (3.5), these exponents do not correspond to the actual behaviour. Thus, even when fluctuation corrections to mean-field theory are taken into account, the actual relation between t and M near the tricritical point is governed by the exponent

$$\beta_t = \beta_0/(1 - \psi_0) = \tfrac{1}{4} \tag{3.37}$$

for $3 < d < 4$. It is interesting to note that the functions (2.12)–(2.15) may be written in terms of either set of exponents (3.4) or (3.36) as, indeed, may the entire classical theory. Thus we have, for example,

$$\begin{aligned} \tau &= t^{\gamma_0}\left(1 + \tfrac{1}{2}(ut^{-\phi_0})(Mt^{-\beta_0})^2 + \frac{1}{4!}(vt^{-\Psi_0})(Mt^{-\beta_0})^4\right) \\ &= t^{\gamma_t}\left(1 + \tfrac{1}{2}(ut^{-\phi_t})(Mt^{-\beta_t})^2 + \frac{1}{4!}v(Mt^{-\beta_t})^4\right). \end{aligned} \tag{3.38}$$

It is straightforward to verify that (3.25) scales with Gaussian exponents, but that if one tries to rewrite it in terms of the correct exponents (3.4) extraneous terms of the form

$$(1/u^*)t^{(1-\varepsilon)/2} \tag{3.39}$$

appear. The solution adopted in I, and first suggested by Sarbach and Fisher (1978a, b), is to introduce an extra variable p which may be thought of as measuring the range of pairwise interactions in an underlying lattice model. In the field theory context, one introduces a factor $p^{-2/d}$, multiplying the gradient term of the Hamiltonian (2.1), the net effect of which is the replacement $u^* \to u^*/p$. In this way, we may formally recover an expression of the form (3.5), except that the scaling function has an additional argument

$$pt^{-\phi_p} \qquad \phi_p = -\tfrac{1}{2}(1-\varepsilon). \tag{3.40}$$

4. Conclusions

For spatial dimensionality in the range $3 < d < 4$, we have shown how to construct crossover scaling functions for thermodynamic quantities which are uniformly valid throughout the tricritical region. To our knowledge, this has been achieved previously only in the spherical model limit, $n \to \infty$, of an n-component model (Sarbach and

Fisher 1978a, b). In this limit, however, classical behaviour is obtained near the wings of the phase diagram, rather than the Ising-like behaviour which we obtain, and which is to be expected for any finite n. Our result for the order-parameter susceptibility is given in (3.25). It is doubtful, however, whether a suitable form of the equation of state can be obtained by our method. The reason for this is that it involves combinations of t, u, v and M which cannot be simply expressed in terms of our basic variables defined in (2.12)–(2.15).

To illustrate our result, we plot contours of the susceptibility near the wing coexistence surface in the (H, t) plane for fixed, negative u, with $\varepsilon = 1$. We first normalise the variables t, M and H according to

$$t' = t/t_c(u, v) \qquad m = M/M_c(u, v) \qquad H' = H/H_c(u, v). \tag{4.1}$$

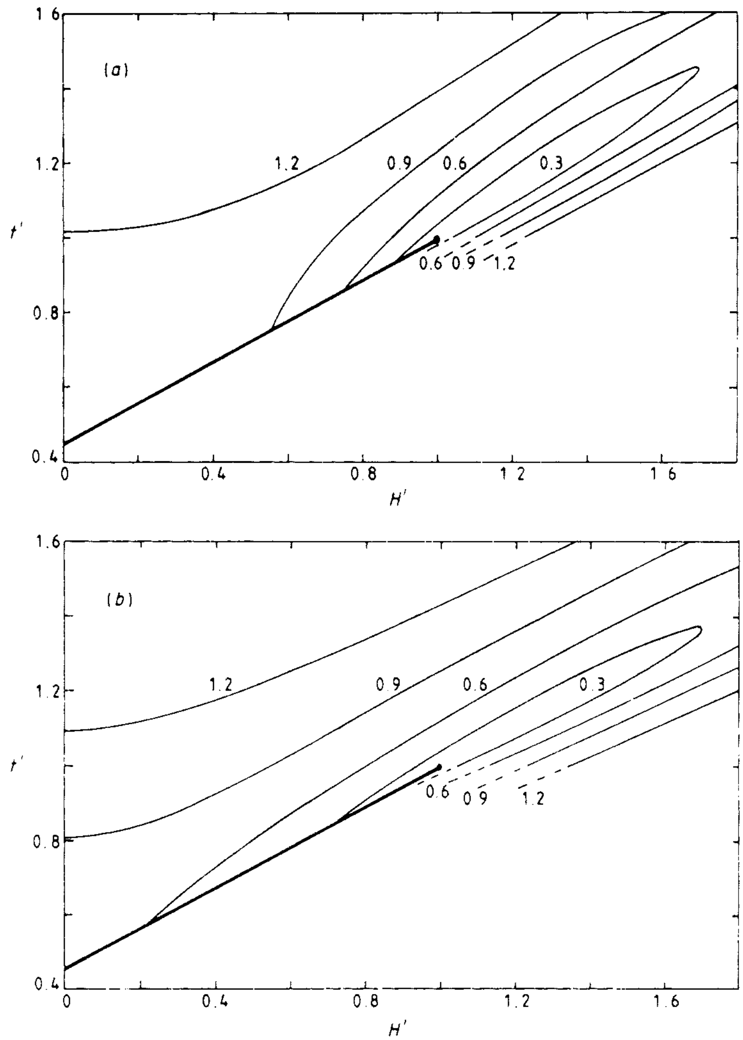

Figure 1. Contours of the inverse susceptibility in the (H', t') plane for negative u calculated at $O(\varepsilon)$ with $\varepsilon = 1$. The bold curve represents the wing coexistence curve for positive H', which terminates at the triple line with $H' = 0$, and at a wing critical point $(H', t') = (1, 1)$. Parameters defined in the text have the values: (*a*) $y = 0.5$, $z = 1$; (*b*) $y = 0$, $z = 0.25$.

When this is done, two additional parameters remain, namely

$$y = (3u^2/2v)^{1/2} \qquad Z = (2v/3)^{1/2}p/u^*. \tag{4.2}$$

In particular, the crossover function X takes the form

$$X^{-1} = 1 - z(\lambda^{-1/2} - y)(1 - 3m^2) \tag{4.3}$$

where λ is determined as the solution of

$$(1 - 3m^2)\lambda = (1 - 3m^2)t' + m^4(5 - 3m^2) - 2m^2(1 - m^2)X^{2/3}. \tag{4.4}$$

As reported in I, and by Sarbach and Fisher (1978a, b) in the many-component limit, the parameter z represents a non-universal dependence of the scaling function on v and p. It seems clear that this apparent non-universality in three dimensions arises from the fact that, while thermodynamic stability requires v to remain strictly positive, the Gaussian fixed point has $v = 0$. This reflects the marginality of the operator ϕ^6 in three dimensions, and one may say that the non-universality reflects the failure of an ε expansion about four dimensions to detect logarithmic corrections to tricritical behaviour in three dimensions (Stephen 1980). In the context of the field theory model we have studied, a fully correct analysis, including both these logarithmic corrections and Ising-like behaviour near the critical loci, would involve simultaneous expansions about three and four dimensions, and this has not yet been achieved.

On the other hand, the parameter y represents power-law corrections to the leading crossover behaviour. In the asymptotic tricritical region, one should set $y = 0$. When this is done, an unexpected singularity occurs in the vicinity of the triple line if z is too large. While this singularity may well be an artefact of our approximations, we note that in the many-component limit (Sarbach and Fisher 1978a, b) there is no tricritical point if z is too large.

For direct comparison with I, we show in figure 1(a) contours obtained for $z = 1$ and $y = 0.5$, where corrections to the asymptotic behaviour remain. In figure 1(b) we show contours obtained for $z = 0.25$ and $y = 0$, which should be representative of behaviour in the asymptotic tricritical region. In contrast with the corresponding figure in I, we observe that the symmetry of the phase diagram is maintained in a perfectly smooth manner. In the absence of a suitable equation of state, values of H' have been obtained by numerical integration of the inverse susceptibility, and for this reason, the region to the right of the coexistence curve is excluded.

Acknowledgment

FWG is grateful to the Science and Engineering Research Council for financial support.

References

Amit D J and Goldschmidt Y 1978 *Ann. Phys., NY* **114** 356–406
Griffiths R B 1970 *Phys. Rev. Lett.* **24** 715–7
Lawrie I D 1979 *J. Phys. A: Math. Gen.* **12** 919–40
Nelson D R and Fisher M E 1975 *Phys. Rev.* B **11** 1030–9
Pfeuty P, Jasnow D and Fisher M E 1974 *Phys. Rev.* B **10** 2088–112
Riedel E K and Wegner F 1972 *Phys. Rev. Lett.* **29** 349–52
Sarbach S and Fisher M E 1978a *J. Appl. Phys.* **49** 1350–52
—— 1978b *Phys. Rev.* B **18** 2350–63
Stephen M J 1980 *J. Phys. C: Solid State Phys.* **13** L83–6